



# Design oriented simulation of contact-friction instabilities in application to realistic brake assemblies

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## Abstract:

*This paper presents advances in non-linear simulations for systems with contact-friction, with an application to brake squeal. A method is proposed to orient component structural modifications from brake assembly simulations in the frequency and time domains. A reduction method implementing explicitly component-wise degrees of freedom at the system level allows quick parametric analyses giving modification clues. The effect of the modification is then validated in the time domain where non-linearities can be fully considered. A reduction method adapted for models showing local non-linearities is purposely presented along with an optimization of a modified non linear Newmark scheme to make such computation possible for industrial models. The paper then illustrates the importance of structural effects in brake squeal, and suggests solutions.*

**Key words:** brake squeal, reduction, component mode, time simulation, instability, contact-friction

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## INTRODUCTION

Automotive brake design is nowadays oriented towards an optimized weight/performance ratio which tends to generate noisy systems. High friction coupling happening at the pad/disc interface is responsible for self-sustained instabilities in the auditive frequency range. The resulting noise can attain 120dB in the brake vicinity and is known as *squeal* between 1 and 16kHz or *moan* under 1kHz.

Unlike the case of low frequency vibrations, the brake performance is not altered by squeal, which happens mostly at low pressure, low speed conditions. The perceived quality is however altered, as the driver's feeling is disturbed and the environmental nuisance is not welcome. Methods

to design silent brakes are mainly empirical and difficult to control, due to modelling issues (*e.g.* contact complexity) or implementation difficulties.

Classical design methods for brake vibration are set in the frequency domain, following experimental results on unstable mode *lock-in* patterns [1] as function of global parameters such as the friction coefficient or the braking pressure. The system is linearized around a working point to apply Lyapunov theorem in which the system stability is related to damping of complex modes. This method shows great limitations and is not sufficient as only potentially unstable modes computed for a fixed contact state are found. In particular, recent work [2] showed that more unstable complex modes can be found than there actually exist.

In the time domain the system with its full non linearities can be simulated, which gives a clear view of brake stability. Such implementation raises many issues as a direct simulation on a full industrial model would require prohibitive computational costs. Contact handling requires relatively small time steps around  $10^{-6}$ s which makes long (100ms) simulations difficult to handle. Using a non linear implicit Newmark scheme on a 600,000 DOF system would actually generate over 1TB of data in over 700 hours.

These issues are dealt with model reduction techniques detailed in section 1.3 and an adaptation of the Newmark scheme to non linear penalized contact vibrations in section 2. Simulation costs thus become affordable yielding from 500MB to 5GB of data in 12 hours.

The computational cost of time simulation is still too important to allow quick parametric studies, besides; from the design point of view, time simulation results must be exploited in a way allowing to close the loop with classical design parameters. It is then necessary to consider components and system characteristics, mostly known in the frequency domain. This paper thus aims at implementing in section 1.2 the Component Mode Tuning (CMT) method presented in [3] to obtain quick design parametric studies from which design directions can be deduced.

A specific treatment of time simulations, based on shape identification presented in section 2.3 allows making the parallel between frequency and time methods. A design method is eventually suggested and illustrated for an industrial brake model provided by Bosch in section 3.

## 1. MODEL REDUCTION STRATEGIES

### 1.1. General concept

Simulation of large industrial models in the design and validation processes requires specific needs which do not match actual computational performance. Indeed, design studies are comfortable with very quick reanalysis simulations from a nominal model, in the hour scale. Validation simulations can take longer, with classical limit of acceptability around 10h (a night).

Although such systems as the industrial brake can be nowadays computed on workstations, and new real eigenvalue solvers like Automated Multi Level Solvers (AMLS) [4] allow to solve systems over a few million DOF, the time required scales

in hours, not in minutes. The full real modes at the nominal state are however accessible, which opens the way to new reduction methods more adapted than the traditional Component Mode Synthesis (CMS) method [5].

CMS was indeed based on the assumption of component independence, static solution capability and explicit boundary coordinates. Interaction information is thus a priori ignored, such that the full finite element basis of their interface is kept. The target application was however different from the aim of this study, which is to reproduce dynamic vibrations of an automotive brake working at a steady state.

Two strategies are presented in the following, which can be combined in the future. Section 1.2 features a reduction method introducing component-wise DOF at the assembly level so that the design loop between structural modification and reanalysis is optimized. Section 1.3 presents a reduction method adapted to models with local non-linearities, which is the case for brake squeal assemblies under the modelling considered for time simulations.

### 1.2. Explicit use of component level DOF

The approach considered here assumes that the components are physically disjoint and that coupling occurs through interfaces that have a physical extent. Since the components are disjoint, the associated DOF  $q_{ci}$  are always distinct. The setup of such model is presented in figure 1.

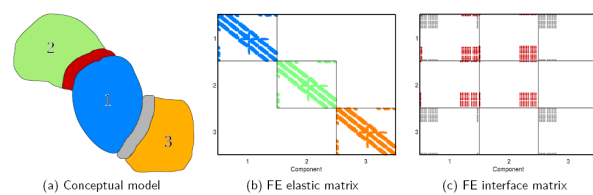


Figure 1. A general mechanical assembly of component with disjoint interfaces and the stiffness matrix, split into a component part and a coupling part

Matrices associated with full or reduced components are non zero for a single component, while interfaces can have non-zero values coupling multiple components. Their diagonal blocks are non null on interface component DOF. In the case of two components one thus has

$$[K] = [K_{el}] + [K_I] = \begin{bmatrix} K_{el1} & 0 \\ 0 & K_{el2} \end{bmatrix} + \begin{bmatrix} K_{I11} & K_{I12} \\ K_{I21} & K_{I22} \end{bmatrix} \quad (1)$$

Figure 1 shows the matrix connectivities for a sample 3 component/2 interface model. The block structure of the component and interface models is indicated through colours. In the application of section 3, the interface matrices correspond to tangent contact/friction coupling models.

Since DOF associated with components are distinct, model reduction is simply performed by generating an assumed basis (classical Rayleigh Ritz reduction)  $T_{ci}$  for each component response, thus yielding a reduction basis  $T_R$  on the whole model

$$T_R = \begin{bmatrix} T_{c1} & 0 \\ 0 & T_{c2} \end{bmatrix} \quad (2)$$

To obtain accurate predictions, one now seeks to retain exact modes for both the system and components. For each component  $i$ , one thus uses a reduction basis that combines component modes in free/free conditions  $\phi_{ci}$  and the trace on the component of exact system modes  $\Phi_{ci}$

$$T_{ci} = \begin{bmatrix} \phi_{ci} & \Phi_{ci} \end{bmatrix}_{Orth.} \quad (3)$$

This method, fully described in [3] allows the generation of very compact systems, as the size of each component block in the reduced matrices, is the size of the number of modes kept in  $T_{ci}$ , that is to say a few hundred. The final brake assembly is thus reduced to around 1,300 DOF.

The reduction basis employed yields a reduced system showing identical dynamics, characterized by the real modes.

The matrix decomposition detailed in equation (1) once reduced on the basis described by equations (2) and (3) shows a very interesting topology, as the elastic matrix is now diagonal, each term corresponding to a component mode frequency. These frequencies are thus explicit in the reduced system and become a design parameter. Since the model is very compact, reanalyses from the reduced model as function of component modes is very quick (less than a minute for real mode computations).

### 1.3. Reduction of models showing local non linearities

Brake squeal dynamics rely on the quality of the pad/disc interface, as it is the location of the main instability. The idea for time simulations is then in a first approach to consider the remaining of the system as linear, based on the pseudo-periodic initial state.

All DOF in the vicinity of this contact area are thus kept explicitly, noted as  $q_c$ . The remaining of the system DOF are noted  $q_i$ .

To achieve the accuracy objective of exact dynamic behaviour, the same method as applied in section 1.2 is employed for the part to reduce, using the trace of the exact real modes of the assembly as assumed shapes. As only the  $q_i$  DOF will be reduced, the trace (or restriction) of the Rayleigh-Ritz basis on this part is only considered. The reduction basis, illustrated in figure 2, is then expressed as

$$\begin{Bmatrix} q_i \\ q_c \end{Bmatrix} = \begin{bmatrix} 0 & [\Phi_{|i} & q_{0|i}] \\ I_c & 0 \end{bmatrix} \begin{Bmatrix} q_R \\ q_c \end{Bmatrix} \quad (4)$$

The pad/disc section is then kept unreduced, while all remaining parts are reduced in a superelement. In the process, the interface DOF are implicitly reduced on the system real modes.

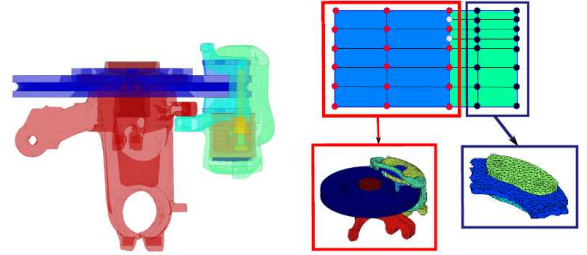


Figure 2 Industrial brake and model reduction strategy for non linear time computations

To cinematically generate a coherent coupling between the finite element part and the superelement, the reduction basis must in addition contain the pseudo-periodic stationary condition  $q_{0i}$ . The final time model features then 30,700 DOF, with a reasonable sparsity. This reduction method was detailed in [6,7].

## 2. TIME SIMULATIONS FOR LARGE INDUSTRIAL SYSTEMS WITH CONTACT-FRICTION

### 2.1. Contact – friction formulations

Contact/friction modelling is commonly split into two formulation strategies, depending on the resolution strategy of the Signorini/Coulomb conditions, giving contact-frictions forces  $f_n$ , and  $f_t$  depending on the friction coefficient  $\mu$ .

Contact can either be formulated as a structural constraint using a Lagrange formulation, or regularized using a penalization formulation. Both formulation strategies are based on the gap  $g$  between a defined set of contact points selected at

the interface, through the notion of gap between two surfaces, which is defined as the relative displacement along the contact normal  $N$  with a possible offset  $g_0$ .

The penalization method chosen here aims at relaxing the contact constraint of Signorini by authorizing a controlled level of interpenetration. Practically, a relationship between the gap and contact forces is established to account for an approached contact constraint.

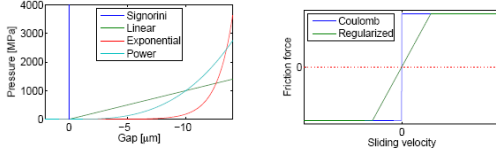


Figure 3. Sample contact (left) and friction (right) laws.

Figure 3 plots several adapted formulations of penalized contact laws. The contact law retained by Bosch is of the exponential type defined at each contact point by

$$p(g) = p_0 e^{-\lambda g} \quad (5)$$

where  $p$  is the contact pressure,  $g$  is the gap,  $p_0$  and  $\lambda$  are parameters to define depending on the interface properties.

Friction implementation follows the definition of the Coulomb law, which for two solids relates the sliding velocity and the friction forces. A basic regularization, plotted in figure 3 and considered in the study, penalizes low sliding velocities (noted  $w_S$ ) through the introduction of a parameter  $k_t$ , such that

$$\begin{cases} f_t = k_t w_S & \text{if } \mu \cdot w_S \leq k_t \cdot f_n \\ f_t = \mu \cdot f_n & \text{else} \end{cases} \quad (6)$$

The pairing strategy and numerical implementation details can be found in [2,6,7,8],

## 2.2. Choice of a time integration scheme

The most common methods come from the Newmark family which implementation is well documented [9]. Simulation of instabilities requires using a well controlled damping strategy [10]. Modal damping is used in the frequency range of interest, which is detailed in section 2.3. Difficult to assess numerical damping implementations are avoided through the choice of an implicit non linear Newmark scheme [9] with scheme coefficients set to  $\beta=0.25$  and  $\gamma=0.5$ .

A non linear Newmark integration scheme is presented in figure 4. The classical method would recompute and factorize the Jacobian  $[J]$  at each time step for each correction increment. The most efficient solvers still need around 10s to factorize a matrix of a size of 30,700, which would take over 600 hours to perform, assuming one correction increment per step.

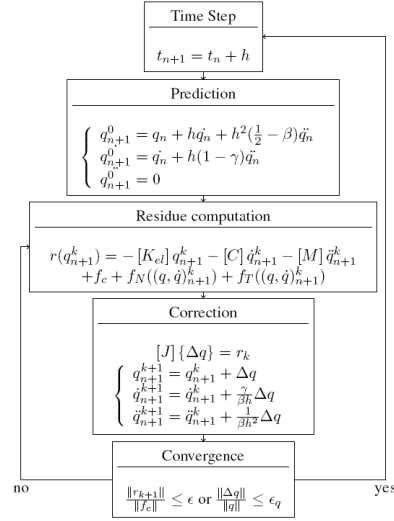


Figure 4. A modified non linear Newmark scheme

The ability to pass such computation in a reasonable amount of CPU time lies in the use of a fixed Jacobian, which is more detailed in [6]. To take contact forces into account, the Jacobian is set up with a mockup linear bilateral contact law which stiffness  $k_c$  is parameterized, so that the Jacobian is written

$$J = \frac{1}{\beta h^2} M + \frac{\gamma}{\beta h} C + K + k_c K_C \quad (7)$$

where  $M$ ,  $C$ ,  $K$  are respectively the mass, damping and stiffness matrices,  $K_C$  a linear coupling matrix of the normal DOF at the contact using a uniform contact law of unitary stiffness.

## 2.3. Design oriented identification of time response

Most common system characterizations are known in the frequency domain, which makes critical the ability to link non linear time simulations of a system to its modal properties - this is especially true for design applications. This section aims at creating this link using the system real modes, which can be used as a parameter through a time domain modal damping strategy or by identification in the response.

The most common and easy to use damping strategy in the frequency domain is modal damping. The concept comes from the assumption [9] that the structure modes are separated enough



not to interact with each other, ensuring a relatively good independence in terms of damping. [7,11] have shown that the modal damping  $C_M$  can be expressed by its application to a velocity vector  $v$  as

$$C_M v = \sum_{k=1}^N (M \phi_k (2\zeta_k \omega_k) (\phi_k^T M v)) \quad (8)$$

which relates first the projection of the velocity on the real mode  $k$  by  $\phi_k^T M v$ , the application of a damping ratio  $2\zeta_k \omega_k$  then the restitution on the finite element basis by the projection  $M \phi_k$ .

This concept can be extended to any shape  $\psi$  defined on the model DOF, provided it is mass normalized:  $\psi^T M \psi = 1$ , which is always possible to obtain.

The modal damping strategy presented is based on the notion of projection of space fields on the system real modes, here the velocity. The same is possible for the displacement so that the projection of the time response on each real mode is possible and the response can be expressed as the sum of each real mode response and an orthogonal part [7]. Modal participations are computed by projecting the response  $q(t)$  as

$$\begin{cases} \alpha_k(t) = \phi_k^T M q(t) \\ \dot{\alpha}_k(t) = \phi_k^T M \dot{q}(t) \end{cases} \quad (9)$$

It is then possible to compute the mechanical energy associated to each real mode  $E_{mk}$  in the time response by computing

$$2E_{mk}(t) = \dot{\alpha}_k^2(t) + \omega_k^2 \alpha_k^2(t) \quad (10)$$

This response allows to evaluate and rank each mode response in the time response thus identifying the main ones. The computation of modal mechanical energies through equation (10) yields the notion of modal sensors.

### 3. INDUSTRIAL APPLICATION

The concepts presented in sections 1 and 2 are here applied to an industrial brake assembly provided by Bosch, presented in figure 2.

The application of the CMT allows to detect *a priori* unstable modes of the system along with their stability sensitivity as function of the component modes. The time simulation allows to detect modes which are the most unstable in realistic condition (*i.e.* regarding the full non linearity). This makes both methods complementary.

A cross application to evaluate the effect of a design change on a component is then suggested. In the following full system modes are designed with the prefix R for real modes and C for complex modes followed by the mode number, no prefix is used for component free/free real modes.

#### 3.1. Stability predictions in the frequency domain

At 12 Bar, the brake system provided features several unstable modes, following the stability diagram of figure 5.

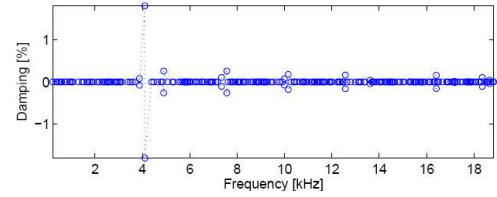


Figure 5. Brake stability diagram at 12 Bar

Some unstable modes can be highlighted, in particular complex modes C44 and C51, which shapes are plotted in figure 6.

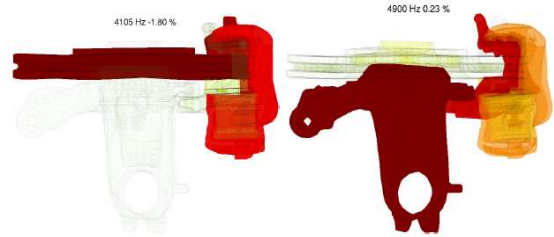


Figure 6. Shapes of unstable complex modes C44 (left) and C51 (right). Colors from blue to red ranking in ascending elastic strain energy per component

From the CMT, the strain energies by components can be extracted, so that the participations can be coarsely observed. Unstable mode C44 shows in particular a disc/outer-pad/caliper interaction.

More detailed information can be obtained by analyzing the sensitivity of the real part of each complex mode to component mode frequencies [3]. Figure 7 plots the sensitivity results for mode C44 shown in figure 6.

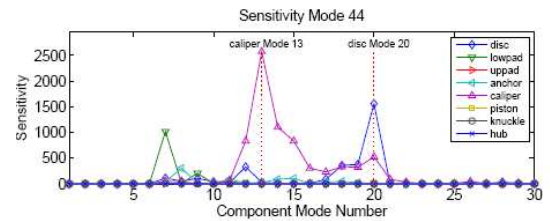


Figure 7. Real part sensitivity ranking for unstable modes C44 as function of the component modes (sensitivity computed for a 5kHz frequency variation)

The sensitivity amplitude shown in figure 7 allows to sort component modes thus obtaining the most sensitive ones. Reanalysis studies are preferred to perturbation studies as they are more precise, but require more computational time. Sensitivity computations then give an *a priori* guess to target specific modes in the analyses performed below.

Focusing on mode C44, it can be seen that the system is mostly sensitive to the caliper. This is a counter-intuitive observation as the friction induced instabilities happen at the pad/disc interface, thus highlighting structural effects.

More detailed reanalyses can then be performed by focusing on the caliper, especially on modes 12 to 14 that seem more relevant.

The shapes of caliper modes 12 to 14 are close, which will make difficult a physical single modification of mode 13. Its effect can however be studied numerically.

Figure 8 shows the evolution of mode C44 as function of caliper modes frequencies (13 or 12 to 14), from -50 to +50%

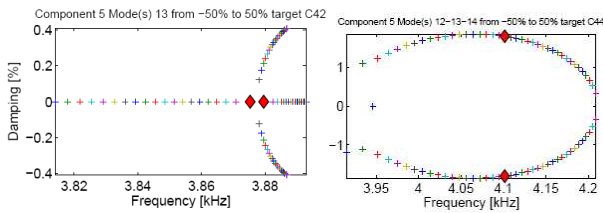


Figure 8. Stability evolution of modes C42 and C44 for caliper modes frequency variations from -50 to +50%.

Left. Caliper mode 13, Right modes 12 to 14.

It can be observed that mode C44 can be stabilized, but the instability is in fact transferred to mode C42, so that modifications of caliper mode 13 alone do not yield a better overall behaviour. Altering caliper modes 12 to 14 does not fully stabilize mode C44 but avoids the instability transfer to mode C42, which makes it a more reasonable solution.

Similar results can be obtained by predicting the effect of a variation of caliper mode damping in the assembly. Figure 9 and 10 show the stability diagrams as function of caliper mode 13 damping from 0 to 50% and of caliper modes 12 to 14 from 0 to 50%.

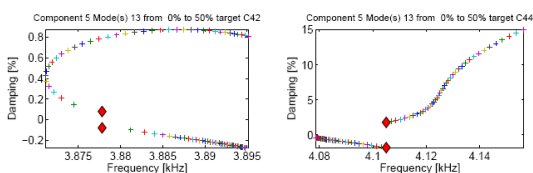


Figure 9. Stability diagram for a variation of caliper mode 13 damping from 0 to 50%.

Altering caliper mode 13 alone destabilizes mode C42 while full stability can be obtained for a given level of caliper mode damping when working on modes 12 to 14, as shown in figure 10.

Care must be taken when trying to stabilize a given mode as destabilization can occur at other frequencies. Stabilizing a specific mode can let another unstable one become predominant in the response.

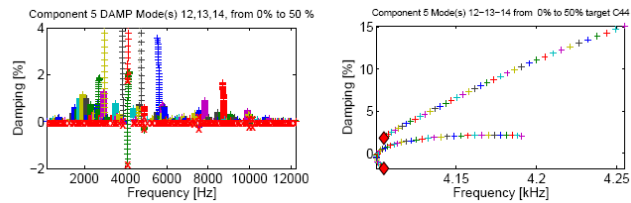


Figure 10. Stability diagram for a variation of caliper modes 12 to 14, damping from 0 to 50%

### 3.2. Squeal simulation

The nominal squeal simulation is performed without damping in the frequency band of interest for 100ms and a time step of  $10^{-6}$ s. Figure 11 shows the simulation result by plotting the braking torque (which is a synthesis of the friction forces).

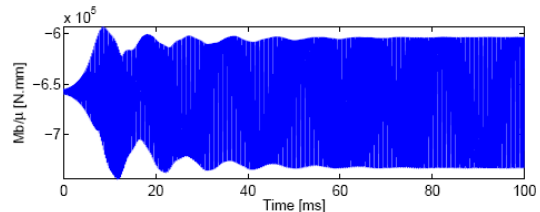


Figure 11. Nominal squeal simulation. Braking torque.

Such simulation is taken as a limit cycle, although no attempt of mathematical characterization is performed. The fact that an exponential divergence occurs after which a saturation pattern appears in which the signal remains bounded is the criterion retained here.

Initialization strategies can be discussed to obtain such cycle in an optimal computation time [8]. The limit cycle obtained is however robust to the initialization strategy, provided an initial perturbation is set.

The use of modal sensors allows to observe the contribution of each real mode to the response in terms of mechanical energy. The study presented in [7] showed that the limit cycles obtained are non linear, so that classical characterizations are difficult. It is in particular interesting to consider new space decompositions, as presented in [7].

### 3.3. Cross study providing design information

Section 3.1 provided design information regarding the brake stability predictions, while section 3.2 briefly illustrated the information that can be output from an actual time simulation. A natural design loop thus appears where rapid predictions can be obtained in the frequency domain with precise component modification indications, which can be validated in the time domain.

The effect of altering caliper modes on the brake assembly has been predicted as efficient to avoid initial instabilities. The use of shape damping in the frequency domain is then used to validate the design direction.

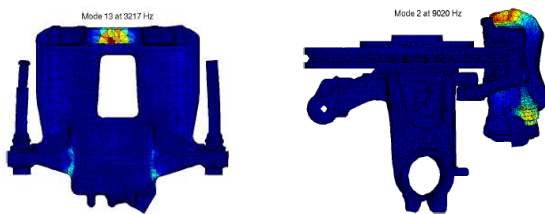


Figure 12. A free/free caliper mode and its projection on the time model. Color code: Left, strain energy. Right, displacement.

Figure 12 presents a caliper mode and its projection on the time model. Since the time model reduction basis does not exactly feature caliper modes, residual displacements at the caliper interface with other components exists. Figure 13 indeed shows the residual displacements on the outer pad.

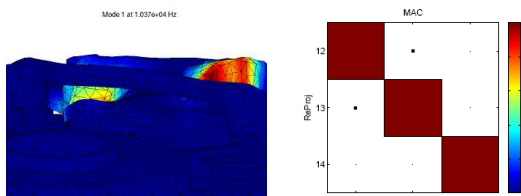


Figure 13. Residual displacement at the caliper/outer-pad interface (right). MAC correlation between the original caliper mode and its restitution from the projection on the time reduction basis (left)

The projection remains however relevant as the caliper mode projected on the time reduction basis and restituted on the full model has perfect MAC correlation with the initial caliper mode shape.

The validation is performed by applying shape damping on the caliper modes considered, starting at the squeal simulation end of figure 11. The results are presented in figure 14 for respectively damping caliper mode 13 and modes 12 to 14 from 2 to 10%.

Damping caliper mode 13 only generates a transient evolution from the limit cycle to another one. Increasing the damping ratio yields another divergence due to an instability transfer to mode C42. This was predicted in section 3.1 and is verified in figure 15, which plots the mechanical energy contributions modes R42 and R45.

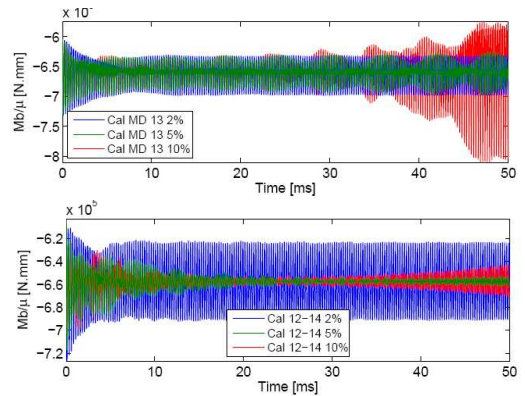


Figure 14. Braking torque results from the initial squeal simulation continued with caliper modes damping 13 (Top) or 12-14 (Bottom)

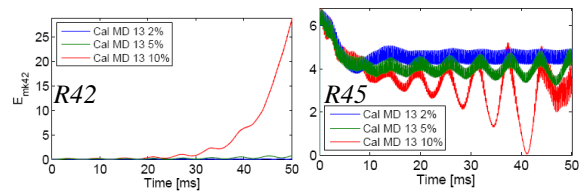


Figure 15. Mechanical energy contribution of modes R42 and R45 when damping caliper mode shape 13

The vibration amplitude is decreased for low damping values, with no stabilization process. For larger damping ratios, the instability is transferred. Adding damping to mechanical assemblies does indeed not necessarily stabilize a system. Such observations were also related in [12]. Figure 16 however shows that damping caliper modes 12 to 14 has a stabilizing effect.

Low values are not sufficient to obtain an actual stability, however, a threshold exists for which the energy associated to unstable mode C44 is dissipated. Another unstable mode then becomes preponderant, as figure 16 shows that mode C51 starts to express itself in the response.

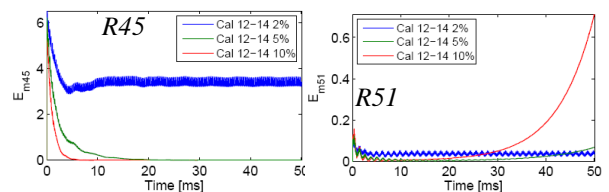


Figure 16. Mechanical energy contribution of modes R45 and R51 when damping caliper mode shapes 12 to 14 from 2 to 10%

Further computations are needed to evaluate actual vibration levels obtained by mode C51 instability regime. A method such as suggested in [8] would also allow some predictions.

## CONCLUSION

The present paper provided a method to perform brake design regarding vibration levels. In the case of brake squeal, instabilities due to the friction coupling at the pad/disc interface generate complex movements at the level of the brake assembly.

The method presented is based on both frequency and time simulations, through different model reduction strategies. The use of the CMT (Component Mode Tuning) method allows generation of very compact tangent models of the order of a thousand DOF allowing very quick parametric studies, in which 20 design points can be computed in half an hour. In addition, the reduction implements component parameters explicitly at the assembly level, so that local modification can be quickly evaluated.

Time simulation of brake squeal are based on a hybrid model, in which most of the brake is reduced on the trace of its real modes, while the pad/disc interface, critical to the instability, is kept unreduced, thus allowing the full expression of the contact-friction non linearity. The optimization performed on the time integration scheme allows to pass a 100ms simulation in 12h for a system of approximately 30,000 DOF. The computation is then performed for validation purposes once design directions have been obtained in the frequency domain by the CMT. Actual vibration level can indeed be obtained in the time domain.

The application considered highlighted the criticality of structural effects in a brake assembly as the caliper was identified as the most sensitive parameter to a pad/disc generated instability.

Further work to the development presented will focus on evaluating design directions more extensively and suggesting a realistic structural modification to stabilize the brake system in practice.

The importance of structural effects is of enough importance that vibrations in the whole assembly are believed to reach levels at which the linearization of other interfaces is arguable. The study of non linear effects at other component interfaces is thus a clear perspective to the work presented.

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