PSEUDO-MODAL REPRESENTATIONS OF LARGE MODELS WITH VISCOELASTIC BEHAVIOR

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Abstract

Damping augmentation materials and devices are often characterized by a frequency dependent complex modulus. The present study focuses on the prediction of the effects of damping treatments on complex structures which can only be modeled with large finite element models. For such models, the inversion of the dynamic stiffness at each frequency point of interest is not a viable approach. Reduction procedures similar to the modal truncation used for elastic structures are thus essential. The paper introduces a robust approach based on a projection on the basis of pseudo-normal modes which correspond to singularities of the conservative part of the dynamic stiffness. The accuracy achieved by this method is demonstrated for a 440 DOF model of a windshield bonded to a rigid frame by a viscoelastic material.

Nomenclature

$[M]$: full-order model mass matrix
$[K]$: elastic part of full-order model stiffness matrix
$[K_v]$: viscoelastic part of full-order model stiffness matrix
E : Young modulus
N : size of the full order model
N_A : number of the considered outputs
N_R : size of the reduced model
N_S : number of the considered inputs
$[T]$: reduction basis
$\{b\}$: frequency-independent input shape matrix
$\{c\}$: frequency-independent output shape matrix
$\{q\}$: degrees of freedom of the full order model
$\{q_R\}$: degrees of freedom of a reduced model
s : Laplace's variable, $i\omega$ ($i^2 = -1$)
$\{u\}$: force inputs
$\{y\}$: displacement outputs
ω_j : normal frequency of the j^{th} mode (rad.s^{-1})
$\{\psi\}_j$: j^{th} mode shape
$[\Psi]$: mode shape matrix

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1 Introduction

The behavior of linear viscoelastic materials is characterized by a complex frequency dependent but linear relation between stress and strain [1, 2]. This linear relation allows the use of finite elements developed for linear elasticity with simple extensions to take into account the complex and frequency dependent constitutive laws.

Viscoelastic materials are often used to enhance the dissipation in complex structures (cars, engine mounts, etc.). These structures are however complex enough to require large order finite element models to get realistic predictions of the dynamic behavior. For simple viscous or histeretic representations of the damping effects, low frequency models of the behavior are typically constructed by projection of the model on a basis of low frequency normal modes and possibly static responses associated to the considered force inputs. Such projections are the basis of *modal analysis* are required to obtain predictions of the low frequency responses of a large model at a reasonable cost.

For general viscoelastic models, normal modes are not defined and projection on modal basis computed for a mean modulus are often inaccurate. The present study extends traditional notions by defining pseudo-normal modes as singularities of the conservative part of the dynamic stiffness and using dynamic corrections computed at the higher end of the model frequency band.

Section 2 summarizes theoretical results linked to modal projections of large order models for the purpose of predicting frequency responses and shows how these results are not applicable to viscoelastic models. Pseudo-normal modes are then defined and their properties illustrated for a simple 2-DOF example.

A method for determining pseudo-normal modes of large order models is described in section 3. The use of pseudo-normal modes for the description of a viscoelastic model is then validated for the case of an elastic windshield bonded to a rigid frame by a viscoelastic material.

2 Modes of viscoelastic models

2.1 Models of viscoelastic structures

The viscoelastic materials considered in this paper are supposed to be described by a complex frequency dependent modulus E as usual in linear viscoelasticity [1, 2]. For the examples considered here, one will only consider a single type of viscoelastic material while the rest of the structure is assumed to be elastic. The general form of the input/output models considered here is thus

$$\begin{array}{c}\left[M\omega^2 + K + E(\omega)K_v\right]\{q\} = [b]\{u\} \\ \{y\} = [c]\{q\}\end{array} \tag{1}$$

where K_v is the stiffness matrix of the viscoelastic part of the structure for a unit modulus. More general viscoelastic materials could be considered but would require more complex representations than (1).

For a standard viscoelastic solid, the Young modulus takes the form:

$$E(\omega) = E_0 \frac{1 + i\omega\beta}{1 + i\omega\alpha} \tag{2}$$

where α is known as the constant of stress relaxation and β is a constant of the model.

Such rational descriptions of the modulus, while allowing the construction of models in a higher dimensional but standard second order form [3], are rapidly limited in their ability to describe experimentally determined moduli over a wide frequency range. In particular the fractional derivative model [4]

$$E(\omega) = E_0 \frac{1 + \sum_{n=1}^{\infty} a_n (i\omega)^{\beta_n}}{1 + \sum_{n=1}^{\infty} b_n (i\omega)^{\alpha_n}} \tag{3}$$

where $0 < \alpha_n < 1$ and $0 < \beta_n < 1$ is not associated to a rational representation and thus cannot be simply treated by standard decompositions on bases of real or complex modes.

2.2 Standard spectral approximations

For large finite element models, one can seldom use the model form (1) to compute time or frequency responses and the traditional approach is to project the model on a truncated basis of normal modes.

For a model with a real and frequency independent stiffness matrix, the normal modes are solution of the eigenvalue problem

$$\left[-M\omega_j^2 + K\right]\phi_j = 0. \tag{4}$$

For M symmetric positive definite and K symmetric positive semi-definite, there are N real modal frequencies ω_j forming a diagonal matrix $\left[{}^{\backslash}\Omega^2_{\backslash}\right]$ and N independent vectors ϕ_j forming a matrix Φ and that these "normal" modes verify two orthogonality conditions

$$[\Phi]^T[M][\Phi] = \left[{}^{\backslash}I_{\backslash}\right] \quad [\Phi]^T[K][\Phi] = \left[{}^{\backslash}\Omega^2_{\backslash}\right] \tag{5}$$

where the modal mass $\{\phi_j\}^T[M]\{\phi_j\}$ is an arbitrary constant which is here set to 1.

The mass and stiffness being diagonal in the basis of the normal modes, the equations linked to the use of modal coordinates are uncoupled for an undamped model. Since each mode is associated with a frequency it is possible to define a number of N_R low frequency modes covering the frequency range for which one seeks a model. A quasi-static approximation of the contribution of other modes is then introduced. Thus, a transfer function between a force, described by the input shape matrix b, and a displacement, described by the output shape matrix c, is approximated by

$$\begin{array}{c} H(\omega) = [c]\left[-M\omega^2 + K\right]^{-1}[b] \approx \\ \sum_{j=1}^{N_R} \frac{[c]\{\phi_j\}\{\phi_j\}^T[b]}{-\omega^2 + \omega_j^2} \\ + \sum_{j=N_R+1}^{N} \frac{[c]\{\phi_j\}\{\phi_j\}^T[b]}{\omega_j^2} \end{array} \tag{6}$$

It is a classical result that this "modal" representation of the transfer function corresponds to a projection of the full order model on the basis generated by the retained normal modes and the static response to the considered load. The assumption is thus that $\{q\} \approx [T]\{q_R\}$ with $[T] = \left[\phi_{j=1,N_R} \quad [K]^{-1}[b]\right]$.

For damped predictions where K is assumed to be complex and possibly frequency dependent. The true spectral decomposition is found by solving the generalized eigenvalue problem

$$\left[M\lambda_j^2 + K(\lambda_j)\right]\psi_j = 0 \tag{7}$$

whose eigenvalues and eigenvectors are complex and lead to a representation of the response of the form

$$[c]\left[-M\omega^2 + K\right]^{-1}[b] \approx \sum_{j=1}^{2N} \frac{[c]\{\psi_j\}\{\psi_j\}^T[b]}{i\omega - \lambda_j} \tag{8}$$

Unlike the case of the "normal" mode decomposition, it is difficult in this form to know how to truncate the series of modal contributions or how to compute a residual contribution for the truncated modes. Furthermore the search for a complex mode solution of (7) requires to have a description of the frequency dependence of K over the full complex plane rather than on the imaginary axis $s = i\omega$.

For models with viscous ($K(s) = K + sC$) or structural ($K(s) = K + iB$) damping, it is relatively common to circumvent the difficulty linked to the computation of complex modes by projecting the damped model on the basis $[T] = \left[\phi_{j=1,N_R} \quad [K]^{-1}[b]\right]$, thus leading to a low order model (as many generalized DOFs as independent columns in the matrix T)

$$\begin{array}{c}\left[-T^T M T \omega^2 + T^T K(\omega) T\right]\{q_R\} = \left[T^T b\right]\{u\} \\ \{y\} = [cT]\{q_R\}\end{array} \tag{9}$$

This projection when applied to a viscously damped model ($K(\omega) = K + i\omega C$) leads to diagonal mass and stiffness matrices (consequence of the orthogonality conditions (5)), and to the assumptions of proportional or modal damping where off-diagonal terms in $T^T C T$ are assumed to be negligible [5, 6].

For cases where the real part of the stiffness is also frequency dependent, the approach proposed in Ref. [7] is to keep a similar projection basis where the modes are computed for the value of the stiffness at one or two frequencies. The application of section 4 will however show that this approach can be strongly dependent on the considered case so that a more robust approach is needed and motivated the introduction of pseudo-normal modes in this paper.

2.3 Pseudo-normal modes and model projection

The pseudo-normal modes of a viscoelastic model are defined as the solutions of the generalized eigenvalue problem

$$\left[-M\omega_j^2 + \mathrm{Re}(K(\omega_j))\right]\tilde{\phi}_j = 0 \tag{10}$$

which corresponds to a straightforward generalization of the standard eigenvalue problem defining normal modes for a frequency independent stiffness. As for the standard problem, these modes allow an accurate representation of the low frequency singularities of $[b]\left[-\omega^2 M + \mathrm{Re}(K(\omega))\right]^{-1}[c]$ (the *conservative* transfer function associated to the real part of the dynamic stiffness). The assumption in using a projection on the low frequency pseudo-modes $\left[\tilde{\phi}_{j=1,N_R}\right]$ is that the changes induced by the imaginary part of the dynamic stiffness will not, for low enough damping, significantly affect the subspace where an approximation of the solution is found.

As for standard spectral approximations, keeping an approximation of the contribution of high frequency modes can be important. It is thus proposed to complement the basis of low frequency pseudo-normal modes by the static response to the load computed for a high frequency modulus

$$[T_A] = \left[K_e + \mathrm{Re}\{E(\omega_{\max})\}K_v\right]^{-1}[b]. \tag{11}$$

Finally the damping effects can be significant so that not taking into account the imaginary part of the dynamic stiffness may limit the achievable accuracy. It is thus proposed to introduce a first order correction to the basis $[T] = \left[\tilde{\phi}_{j=1,N_R} \quad T_A\right]$ by computing the static response to the load generated by the imaginary part of the stiffness when exciting a given pseudo-normal mode

$$[T_{Cj}] = \left[K_e + \mathrm{Re}\{E(\omega_j)\}K_v\right]^{-1}[K_v]\left\{\tilde{\phi}_j\right\}. \tag{12}$$

The basis $\left[\tilde{\phi}_{j=1,N_R} \quad T_{Cj=1,N_R} \quad T_A\right]$ is contains twice the number of pseudo-normal modes in the considered band but the accuracy improvement (see section 4) may well justify the additional cost.

2.4 Pseudo-normal modes for a 2-DOF example

Let us consider the two spring-two mass system shown in the figure 1 where the stiffness of the first spring follows the 3-parameter viscoelastic law

$$k_1(\omega) = k_0 \frac{1 + i\omega\beta}{1 + i\omega\alpha} \tag{13}$$

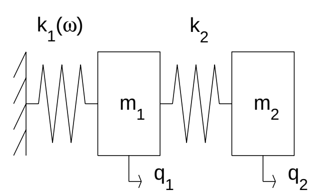

Figure 1: 2-DOF viscoelastic model

Applying definition (10), the pseudo-normal modes are solutions of

$$\left[-M\omega^2 + K + \tfrac{1+\omega^2\alpha\beta}{1+\omega^2\alpha^2}K_v\right]\left\{\tilde{\phi}_j\right\} = [0] \tag{14}$$

which is equivalent to the second order eigenvalue problem in ω_j^2

$$\left[-\alpha^2 M\omega^4 + A\omega^2 + K_e + K_v\right]\{\phi\} = [0] \tag{15}$$

with $A = \alpha^2 K_e + \alpha\beta K_v - M$.

An exact solution of (15) is classically found (see Ref. [3] in particular) by transformation to a first order generalized eigenvalue problem in ω_j^2

$$\begin{bmatrix} A & \alpha^2 M \\ \alpha^2 M & 0 \end{bmatrix} \left[\backslash \Omega^2_\backslash\right] [\theta] + \begin{bmatrix} K_e + K_v & 0 \\ 0 & -\alpha^2 M \end{bmatrix} [\theta] = [0] \quad (16)$$

where $[\theta]_{2N\times 2N} = \begin{bmatrix} [\Phi] \\ -[\Phi]\left[\backslash \Omega^2_\backslash\right] \end{bmatrix}$.

The pseudo-modes are the solutions of (15) associated with real ω_j^2. While the existence of such real eigenvalues is not demonstrated here, they exist in practice and it is clear that the associated deformations (pseudo-modes) are real valued.

For $m_1 = m_2 = 1$, $k_0 = k_2 = 3$,$\alpha = 2/3$ and $\beta = 1$, the spring-mass system has two pseudo-modes which when mass-normalized to 1 are

$$\begin{aligned} \omega_1 &= 1.135 \quad \tilde{\phi}_1 = \begin{Bmatrix} 0.495 \\ 0.869 \end{Bmatrix} \\ \omega_2 &= 2.959 \quad \tilde{\phi}_2 = \begin{Bmatrix} 0.887 \\ -0.462 \end{Bmatrix} \end{aligned} \quad (17)$$

The pseudo-modes being defined by the eigenvalue problem (16), they do not diagonalize the M, K_e and K_v matrices. Here for example, the mass orthonormality test leads to

$$[\Phi]^T [M][\Phi] = \begin{bmatrix} 1 & 0.038 \\ 0.038 & 1 \end{bmatrix} \quad (18)$$

where one notes significant off-diagonal terms. The stiffness orthogonality tests evaluated at the pseudo-mode frequencies

$$\begin{aligned} [\Phi]^T [K(\omega_1)][\Phi] &= \begin{bmatrix} 1.288 & 0.049 \\ 0.049 & 8.248 \end{bmatrix} \\ [\Phi]^T [K(\omega_2)][\Phi] &= \begin{bmatrix} 1.447 & 0.334 \\ 0.334 & 8.757 \end{bmatrix} \end{aligned} \quad (19)$$

do not indicate any orthogonality even though the condition $\tilde{\phi}_j^T K(\omega_j)\tilde{\phi}_j = \omega_j^2$ remains true for mass normalize pseudo-modes.

The final objective being to compute FRFs, one will consider the response near the frequency of the first pseudo-mode. One will compare projections of the 2-DOF model on the first pseudo-normal mode $\tilde{\phi}_1$ and the first normal normal mode associated to the low frequency stiffness (setting $k_1 = k_0$ the first mode $\phi_1 = [0.526 \;\; 0.851]^T$ is found at $\omega_1 = 1.070$ which should be compared with the pseudo-normal mode given in (17)).

For a force applied on mass 2 and a response measured at the same location ($b^T = c = \begin{bmatrix} 0 & 1 \end{bmatrix}$), figure 2.4 shows the conservative response (obtained by setting the imaginary part of k_1 to zero) and the damped response. The conservative response clearly indicates that only the pseudo-mode shape results in a good placement of the first resonance. Obtaining a good representation of singularities of the conservative response is really the basis for the definition of pseudo-modes.

In general this approximation is sufficient to represent correctly the damped response (as shown in the figure) but improvements can be found using the correction proposed in eq. (12).

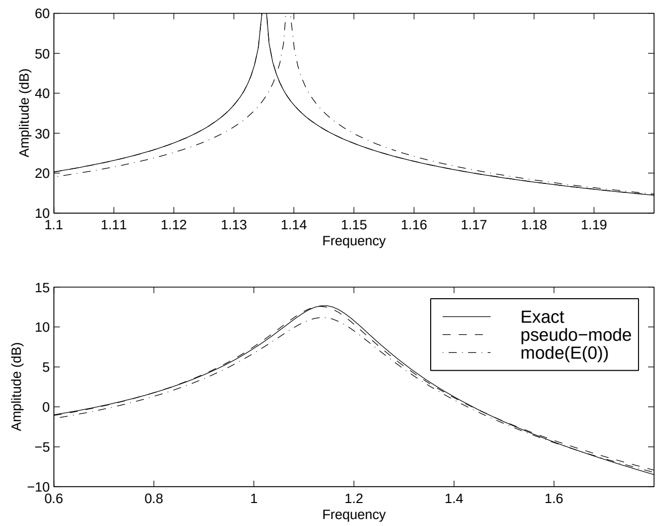

Figure 2: a) conservative response (i.e. with the imaginary part of k_1 set to zero) b) damped response

3 Determination of pseudo-normal modes

The pseudo-normal modes must be determined using a iterative method allowing the determination of those associated with the lowest frequencies. The principle of such a method is to use a variable projection basis T^k that is adapted as more pseudo-modes are determined. Methods to build the variable projection are those traditionally used to estimate low frequency modes of frequency independent models (Lanczos or subspace-iteration).

The test used for the determination of a pseudo-normal mode at $om\tilde{e}ga_j$ is the existence of a normal mode solution of

$$\left[\mathrm{Re}(K(\tilde{\omega}_j)) - \omega^2 M\right]\{\phi\} = \{0\} \quad (20)$$

whose frequency is equal to $\tilde{\omega}_j$. The pseudo-normal mode shape is then given by the normal mode shape found in (20).

The pseudo-normal frequencies and modes are computed by frequency bands. Starting at 0, one computes the normal modes associated with $\left[\mathrm{Re}(K(0)) - \omega^2 M\right]\{\phi\} = \{0\}$. The first frequency ω_1 of this model is taken as an estimate of the first pseudo-normal mode frequency ($\tilde{\omega}_j^m$ with $j = m = 1$). The first considered projection basis T^1 will contain the first N_R normal modes associated to $K(\tilde{\omega}_1^k)$.

In this basis one will search for roots of the projected dynamic stiffness. One will thus solve

$$\det\left(\left[T^{(k)}\right]^T \left[\mathrm{Re}(K(\omega)) - \omega^2 M\right]\left[T^{(k)}\right]\right) = 0 \qquad (21)$$

A first approximation of these roots is given by the normal mode frequencies associated with $K(\omega_j^k)$. This approximation is used to select a frequency range where a low order polynomial approximation is built leading to a rapid and robust convergence to the actual roots of (21).

The next projection basis T^{k+1} contains the N_R normal modes associated with $K(\tilde{\omega}_j^m)$ (the current estimate of the next pseudo-normal mode frequency). One then iterates on the determinant search until $\tilde{\omega}_j^m$ and $\tilde{\omega}_j^{m+1}$ differ by less than a given tolerance. The condition (20) is then taken to be verified and the associated normal mode used as the estimate of the pseudo-normal mode shape. One then looks for the next pseudo-mode (of index $j+1$).

In practice, and exact estimate of the N_R normal modes associated with $K(\omega_j^{\tilde{m}})$ would be too costly and one uses an iterative correction of the previous estimate of those modes in a procedure similar to the one proposed in Ref. [8].

4 Application

To validate the proposed approaches, a rectangular $0.35m \times 0.55m \times 4mm$ glass plate representing a windshield will be considered. The four edges of this plate are bonded to a rigid frame using a $5mm \times 0.05mm$ ribbon of viscoelastic material. For the purpose of this example, the properties of the ISD112 [9] will be used. The frequency dependence of the properties of this material are shown in figure 3. One notes the increase of the storage modulus by a factor higher than 4 over the considered frequency range and decreasing loss factor. Additional properties of the viscoelastic are $\nu = 4.5$ and $\rho = 1.2 kg/m^3$. The elastic properties of the glass plate are $E = 6\ 10^{10} N/m^2$, $\nu = 2.5$, $\rho = 2.5\ 10^3 kg/m^3$.

The windshield is modeled using a 7 by 10 grid of 4-node/20-DOF thin plate element while the bonding uses 30 8-node/24-DOF solid elements. The model thus as 440 DOF while the reduced models considered will have 22 (pseudo-mode and multi-model) and 43 vectors (pseudo-mode with first order correction).

The elastic properties of the glass plate are $E = 6\ 10^{10} N/m^2$, $\nu = 2.5$, $\rho = 2.5\ 10^3 kg/m^3$.

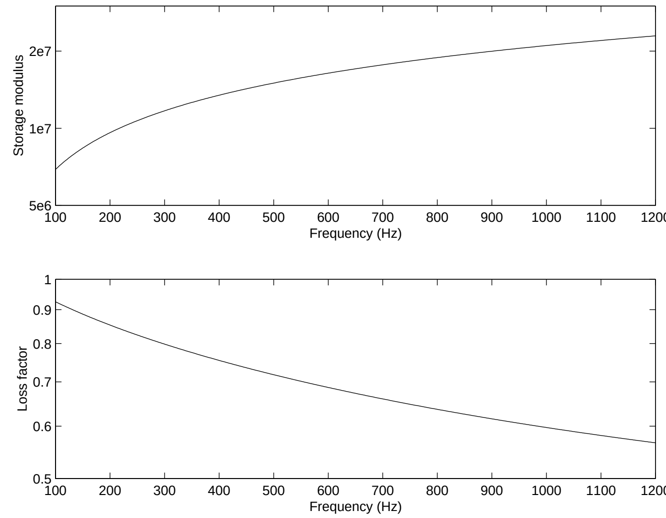

Figure 3: Frequency-dependent modulus characteristics

In ref. [7], it was proposed to use a multi-model approach combining modal bases associated with the modulus at different frequencies. A straightforward application of this approach would be too retain the first 11 normal modes of a model with a low frequency modulus (first half of the considered frequency band), and the normal modes 12 to 21 of a model with a high frequency modulus (second half of the frequency band). The alternative proposed here is to use the pseudo-normal modes computed in the frequency range of interest (0-1200 Hz). As for standard spectral approximations, these two bases are complemented by the static response to the considered load (11).

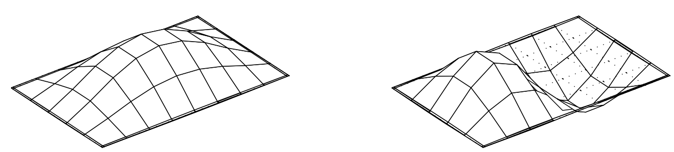

Figure 4: Pseudo-normal modes

To understand the motivation for pseudo-normal modes, one first predicts the frequency response assuming that the imaginary part of the modulus is equal to zero (which corresponds to the extension of the notion of conservative system associated to a model). Figure 5 clearly indicates a very good accuracy of the pseudo-normal mode projection (minor differences only visible near anti-resonances) whereas the multi-model reduction leads to mismatches of the resonances.

Further analysis would actually show that keeping the normal modes associated with the high frequency modulus lead to good predictions but assessing this accuracy without computing the exact results would not be possible. The pseudo-normal modes are thus a more robust approach to building an accurate projection of a viscoelastic model.

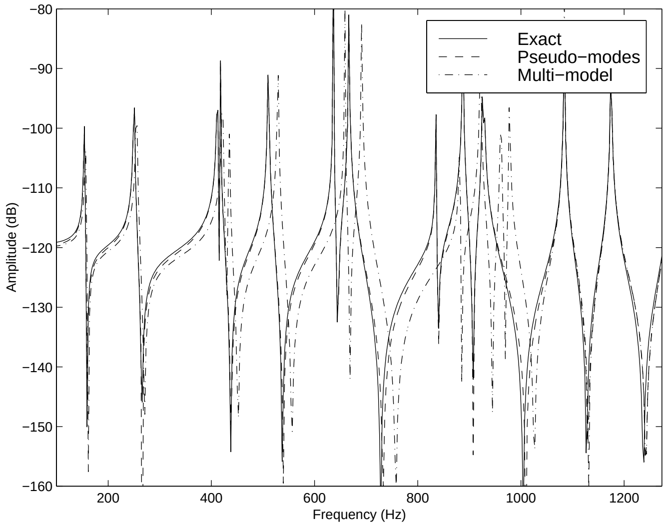

Figure 5: Frequency response functions: (—) exact response , (- -) pseudo-normal modes, $(\cdot - \cdot)$ multi-model

In figure 6, the damped predictions are compared with the exact response for the same two reduction bases. The poor accuracy of the chosen multi-model reduction is again clearly apparent while the pseudo-normal modes give very good although not perfect results.

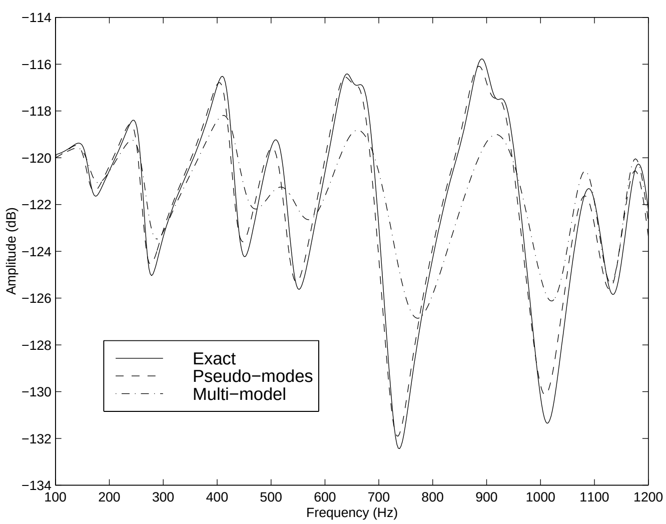

Figure 6: Frequency response prediction: (—) exact response , (- -) pseudo-normal modes, $(\cdot - \cdot)$ multi-model

If the accuracy obtained with the simple pseudo-normal mode basis, significant improvements can be achieved with the first order correction (12). In the present example. The difference between the true response and the corrected pseudo-normal mode model is not visible in figure 7a and figure 7b shows that the relative error is almost always below 10^{-3}.

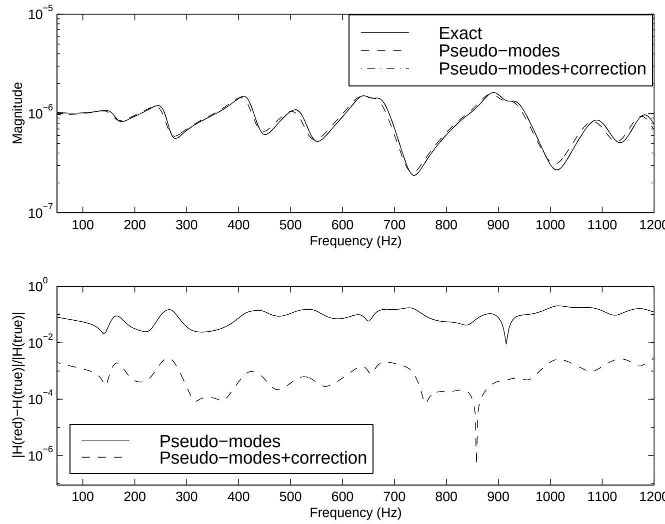

Figure 7: (a) Frequency response function: (—) exact response , (- -) pseudo-normal modes, $(\cdot - \cdot)$ pseudo-normal modes with first order correction, (b) Quality of the frequency response predictions (–) pseudo-normal modes, (- -) pseudo-normal modes with first order correction

5 Conclusions

Pseudo-normal modes and the proposed first order correction for the damping effects give a robust and well defined procedure to build projection bases allowing the prediction of the low frequency response of linear viscoelastic structures. This approach extends modal analysis procedures to a large class of viscoelastic structures even when no analytic expression of the modulus exists.

For very large models, the simple approach of inverting the dynamic stiffness at each frequency point is not viable [10]. The proposed approaches are thus essential to allow the prediction of frequency response functions for large viscoelastic models. Improvements are however still needed in the algorithm used to compute pseudo-normal modes. Finally, the methods to build equivalent mass, viscous-damping, stiffness models in the modal domain proposed in Ref. [7] should be tested in this new framework.

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