Use of Generalized Interface Degrees of Freedom in Component Mode Synthesis.

Etienne Balmès (DRET A/C)
ONERA, Structures Direction

ABSTRACT

Substructuring, Component Mode Synthesis, super-element, and related methods are used in numerous applications. The principle of these methods is to represent the model of a system by coupled component models. In most cases, prior knowledge of the predictions of interest is used to reduce component models. Continuous interfaces between plate or solid components traditionally imply the use of large component models (with more degrees of freedom than there are in the interfaces). This strongly limits the interest of using reduced models. An automated, and yet computationally robust and efficient, treatment of component coupling conditions is introduced. This approach allows significant extensions to traditional component model reduction but does not eliminate the risk of poor predictions linked to locking of incompatible component models. The example of a stiffened panel is used to demonstrate the practicality of the proposed framework. In particular, methods for the reduction of plate interface models and for the treatment of locking are addressed.

NOMENCLATURE

The paper uses the standard *Modal Analysis* notations. Variables not contained in the standard are

b, c	input and output shape matrices
u, y	vectors of inputs, outputs
$K = Ms^2 + Cs + K$	dynamic stiffness matrix
λ	Lagrange multipliers linked to interface forces
$s = i\omega$	Laplace variable

1. INTRODUCTION

Model reduction procedures have the general objective of using solutions of intermediate problems to later allow multiple low cost predictions of the global response. A large class of methods (including substructuring, Component Mode Synthesis (CMS) [1], and super-element methods) decompose systems into components and perform intermediate computations at the component level. Approximate component models are generally used, but recently the availability of parallel computers has motivated the use of substructuring methods to compute exact solutions (one then usually talks of domain decomposition methods).

A major difficulty is that the reduction of component models and the enforcement of component coupling conditions cannot be considered separately. In particular, this has traditionally led to the use of nodal Degrees Of Freedom (DOF) for the representation of component interfaces. For interfaces between plates or solids, the resulting number of DOFs and the associated computational costs are strong limitations for the use of substructuring methods. Methods such as those proposed in this paper can thus significantly extend the applicability of traditional methods.

It is shown in section 2 that component coupling conditions can be treated as generalized kinematic (for displacement) or natural (for force) boundary conditions. For compatible and incompatible meshes and for nodal and integral [2] formulations, boundary conditions on discretized models take the form of a finite number of constraints which can be treated through a direct elimination or the addition of Lagrange multipliers. A computationally efficient and robust direct elimination algorithm is discussed.

Traditional reduction methods and extensions allowing reduced representations of interface deformations are discussed in section 3. With the direct elimination approach, reduced component models can be constructed using any method. However, arbitrary reduction procedures often lead to incompatible models which can give poor predictions due to a locking phenomenon which is addressed.

The proposed generalizations of the treatment of interface boundary conditions and component model reductions, give a very general framework to analyze a wide range of new and existing CMS methods. Typical foreseen applications are illustrated in section 4 using the example of a stiffened panel with a hole. It is first shown how reduced models with much fewer DOFs than interface DOFs can be created and used to provide accurate predictions of the coupled response. It is then shown how the locking of incompatible component models imposes trade-offs between global accuracy, interface continuity and reduction without prior knowledge of other components.

2. COUPLING OF COMPONENT MODELS USING GENERALIZED BOUNDARY CONDITIONS

2.1. Properties of model form

This study considers cases where all substructures have an accurate second order representation, generally constructed using the finite element method, of the form

$$\begin{aligned}\left[Ms^2 + Cs + K\right]\{q\} &= [b]\,\{u\} \\ \{y\} &= [c]\,\{q\}\end{aligned} \qquad (1)$$

In these models, the response is fully described by a finite number of *degrees of freedom* (DOFs) q that depend on time/frequency. The dynamic stiffness matrix $K = Ms^2 + Cs + K$ gives the relation between the response of the model DOFs q and the model loads F_q. (The dynamic stiffness is assumed symmetric but extensions to non symmetric cases are possible).

Physical displacements (translations, rotations, stresses, strains, etc.) are called *outputs* y and assumed to be linearly related to the DOFs q through output shape matrices c ($y = c\,\{q\}$). For example, the matrix c associated with displacement outputs of a displacement based finite element corresponds to the evaluation of the element shape functions at the considered node.

Similarly loads (applied forces, aerodynamic or acoustic pressure fields, control forces, gravity, etc.) are represented by the product of time independent input shape matrices b and time/frequency dependent *inputs* u ($F_q(u) = b\,u$).

For many applications one is interested in coupling models of different components. This will be done here by appending models of different components

Proceedings of IMAC, February 1996.

$$\begin{bmatrix} K_1(s) & \\ & K_2(s) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \{u(s)\}$$
$$\{y(s)\} = [c_1 \quad c_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{2}$$

and considering generalized boundary conditions (i.e. constraints) linked to the connection of the different components.

Models (1) and (2) only differ by the fact that in (2) component indices are used to characterize blocks of degrees of freedom. In the following analysis one will assume that the models of all components have been appended (as in (2)) to form a single uncoupled model of the form (1) (the handling of blocks of DOF is a computer related issue and its use in analytical presentations of methods leads to notations that are more complicated than necessary).

2.2. Coupling of component models using generalized kinematic boundary conditions

Predictions of the system response are obtained by coupling component models. For the underlying continuous model, component coupling is traditionally achieved by enforcing continuity of displacements y (in general translations along three axes, but possibly rotations, temperature, pressure, etc.) on the boundary

$$\Delta y(x) = (y_1(x) - y_2(x)) = 0 \quad \text{for all } x \in \partial\Omega \tag{3}$$

It clearly appears in (3) that the continuity requirement of interface displacements corresponds to a generalized kinematic (also named Dirichlet or essential) boundary condition (Δy is set to 0). In a Ritz type analysis, trial functions only need to verify kinematic boundary conditions [3]. Natural (also named Neumann) boundary conditions may also be enforced but they correspond to further model reduction and induce a loss of accuracy (see details in section 2.3).

For a discretized model (an element, a group of elements, the reduced model of a group of elements, etc.), interface displacements are represented by a linear combination of shape functions so that for any position x one can construct an output shape matrix $c(x)$ such that the deformation described in the model DOFs by the vector q corresponds to a displacement

$$\Delta y(x) = [c(x)] \ \{q\} \tag{4}$$

where $c(x)$ is generally sparse because DOFs of different components have been appended in a single vector q (Eq. (2)) and some DOFs do not induce interface deformations.

Result 1: for a discretized model, generalized kinematic boundary conditions of the form (3) can be expressed in the form of a finite set of constraints on relative interface displacements Δy_{Int}

$$\{\Delta y_{Int}\} = [c_{Int}] \ \{q\} = 0 \tag{5}$$

This result simply comes from the fact that, for a finite model, the range of $c(x)$ in the space of all functions of x is finite. If Δy represents the relative displacement of the edges of two contiguous compatible elements the constraints c_{Int} simply correspond to the equality of corresponding nodal DOFs (one generally calls compatible or conforming, elements that are such that equality corresponding nodal DOFs is equivalent to the continuity of shape functions at the interface). The generalization of this approach to compatible component models is straightforward (just impose the equality of corresponding DOFs of the interfaces, case A in Fig. 1).

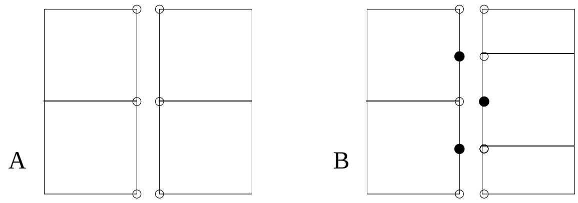

Fig. 1: Continuity constraints c_{Int} can be built by enforcing continuity of displacements at nodes of the elements ○ as well as intermediate nodes ●.

For incompatible interfaces the determination of the constraints may pose some difficulties. The easiest approach is to use a set of nodal constraints such as taking all nodes of elements on both sides of an interface (case B in Fig. 1). For cases with intermediate nodes and elements with non-nodal DOFs, a knowledge of the element shape functions is needed to relate the nodal displacements to the element DOFs but this is not a fundamental problem.

Different authors (e.g. Ref. [2]) have used discretizations of the interface model to build a finite vector space V_λ of λ functions (representing interface forces) and integral constraints to impose the boundary condition

$$\int_\Gamma \lambda \Delta y \, d\Gamma \quad \text{for all } \lambda \in V_\lambda \tag{6}$$

For Δy and λ taken in finite vector spaces, this clearly leads to a finite set of conditions of the form (5). For discretized models, full sets of nodal or integral constraints are fundamentally equivalent and their use should be mostly motivated by implementation considerations.

In cases with incompatible meshes, incompatible models, or points common to more than two substructures, it is difficult to define a minimal set of independent constraints leading to exact verification of the boundary conditions. In such cases, it is preferable to consider obviously redundant sets of constraints and take proper care of the fact that some of these constraints may not be independent (see section 2.4).

For certain component models, the use of redundant constraints may lead to poor solutions (an over-stiffening effect known in finite element analysis as locking). Relations between global accuracy, selected constraints and component model reduction are addressed in sections 2.4 and 4.4.

2.3. Model reduction using generalized natural boundary conditions

In many applications, it is known that no external forces are applied to a given set of DOFs. For a discrete model, such natural (also called Neumann) boundary conditions can be written

$$[c_{Int}][Ms^2 + Cs + K]\{q(s)\} = 0 \tag{7}$$

For most finite elements, continuity of stresses across boundaries or nullity on edges is either not imposed or imposed in a weak sense. As a result, the link between continuous and discrete natural boundary conditions is not obvious and may lead to interesting results [4].

If the DOFs q are chosen so that c_{Int} just corresponds to the extraction of the first C rows and A refers to the other rows. Condition (7) is clearly equivalent to limiting q to the subspace defined by

$$\begin{Bmatrix} q_A \\ q_C \end{Bmatrix} = \begin{bmatrix} I \\ -\left(M_{CC}s^2 + C_{CC}s + K_{CC}\right)^{-1}\left(M_{CA}s^2 + C_{CA}s + K_{CA}\right) \end{bmatrix} \{q_A\} \tag{8}$$

The solution (8) for an exact natural boundary condition (7) is non-linear in frequency which makes it impractical for most uses. Several methods have thus been introduced to obtain approximations of the solution (8). The simplest idea is to find a first order expansion near $s=0$

$$\begin{Bmatrix} q_A \\ q_C \end{Bmatrix} = \begin{bmatrix} I \\ -K_{CC}^{-1}K_{CA} \end{bmatrix} \{q_A\} \tag{9}$$

where all knowledgeable readers will recognize a static or Guyan condensation [5]. The fact that this approach gives statically exact results is a distinct advantage.

Another approach is to evaluate condition (8) at another frequency that is closer to the frequency of interest (one then talks of dynamic condensation [6]). Finally, one can also consider [7,8] MacLaurin series expansions of the inverse

$$\left(M_{CC}s^2 + K_{CC}\right)^{-1} = K_{CC}^{-1} - \left(K_{CC}^{-1}M_{CC}K_{CC}^{-1}\right)s^2 + h.o.t. \tag{10}$$

For CMS methods, many authors have considered free vibration problems where no external forces are applied on the interface. For compatible substructure models, the absence of external forces on the interface is clearly equivalent to the equilibrium of internal forces applied on each substructure. Different approximations of a generalized natural boundary condition have thus been considered to condense boundary degrees of freedom [7-9].

2.4. Numerical methods for the handling of constraints

It appears from previous sections that all the coupled component models of interest can be written in the form (this was rapidly recognized [8] in studies on CMS methods)

$$\begin{gathered} \left[Ms^2 + Cs + K\right]\{q(s)\} = [b]\,\{u(s)\} \\ \{y(s)\} = [c]\,\{q(s)\} \end{gathered} \quad \text{with} \quad \left[c_{Int}\right]\{q(s)\} = 0 \tag{11}$$

By using discrete Lagrange multipliers for the constraints [8,10], one finds the usual form of mixed finite element formulations [11]

$$\begin{bmatrix} K(s) & c_{Int}^T \\ c_{Int} & 0 \end{bmatrix} \begin{Bmatrix} q \\ \lambda \end{Bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{12}$$

The problem can be solved directly using a model of the form (12). This may be particularly efficient for implementations on parallel computers, as some steps can be performed on a substructure by substructure basis [12,10]. The classical approach [1,13] is however a direct elimination of constraints. This approach consists in determining a basis T for the null space of the constraints $(\mathrm{range}(T) = \ker(c_{Int}))$ and projecting the model on this subspace

$$\begin{gathered} \left[T^TMTs^2 + T^TCTs + T^TKT\right]\{q_R\} = \left[T^Tb\right]\{u\} \\ \{y\} = [cT]\,\{q_R\} \end{gathered} \tag{13}$$

In many cases the determination of T is obvious. For example, the standard assembly procedure for finite element models takes into account that DOFs of corresponding nodes for contiguous elements have the same displacement by only defining one set of nodal displacements and projects the model by adding the contributions of the different elements connected to a given DOF.

To treat general classes of problems, it is proposed to use a fully automated algorithm for the determination of the null space of c_{Int}. A numerically efficient and robust algorithm was constructed [14] by taking the following factors into account.

As pointed out in section 2.2, simplicity in the application of substructuring methods often leads to sets where all constraints are not independent. A way to distinguish the effective rank of the matrix is thus needed. The singular value decomposition or the related QR decomposition are known to provide very robust measures of effective rank and are thus quite appropriate [12]. Plate or solid problems do however lead to large numbers of interface degrees of freedom so that the cost of such decomposition may become excessive. The size of pivots in an LU decomposition provides a slightly less robust but significantly cheaper approach which has shown good results on a variety of cases [14].

In many cases and particularly when using nodal constraints, c_{Int} can be decomposed in blocks of constraints that only affect certain degrees of freedom. By automatically monitoring the existence of such blocks it is possible to consider a single set of constraints for all substructures and yet retain the low cost of finding the null space for blocks of constraints.

Many CMS approach are constructed to take advantage of particular forms of the constraint equations and thus eliminate them easily. The general approach proposed here loses such simplifications but gains the possibility of coupling components reduced with arbitrary procedures and of using redundant constraints. Furthermore, considering that extremely efficient procedures can be constructed to determine the null space, the overall computational advantage of using an *ad hoc* elimination procedure is often small.

The proposed automated procedure gives coupled predictions for arbitrary reduced component models. For incompatible component models, locking may however lead to very poor results (see more details in sections 3.3 and 4.4).

3. RELATIONS BETWEEN REDUCTION AND COUPLING OF COMPONENT MODELS

3.1. Displacement based reduction of component models

The construction of many finite elements can be seen as the computation of a number of particular solutions to continuous analytical problems and the projection of the continuous equations on the associated basis. Similarly, super-elements or reduced component models correspond the projection of a component model onto a basis of particular solutions defined for the component DOFs. Such an approach corresponds to a Rayleigh Ritz analysis, or *displacement based reduction approaches*, where one seeks approximate solutions in a reduced subspace corresponding to the range (described by reduced DOFs q_R) of a rectangular matrix T

$$\{q_{\mathrm{True}}\} \approx [T]\,\{q_R\} \tag{14}$$

The validity of the projection is based on the assumption, that all effectively found displacements q of the full order model have a close approximation in the range of T. The projection (14) applied to loads and displacement of the full order model (1) lead to the reduced model (with n_{QR} rather than n_Q DOFs)

$$\begin{gathered} \left[T^TMTs^2 + T^TCTs + T^TKT\right]\{q_R\} = \left[T^Tb\right]\{u\} \\ \{y\} = [cT]\,\{q_R\} \end{gathered} \tag{15}$$

Models are used to compute “qualities” which characterize the system response. Typical “qualities” are static responses to fixed loads, stress/strain distributions, modal frequencies, modeshapes, or damped system responses. Full (1) and reduced (15) models can give estimates of the same “qualities”. The difficulty is to choose the reduction basis T so that, for the qualities of interest, predictions of the full and reduced models are similar.

Published ways to construct reduction bases are so numerous that the search for an exhaustive listing would be futile. In almost all cases however, the idea is to use solutions to representative sub-problems that allow a good representation of external and inertial loads.

All models are only valid in certain conditions. Model reduction procedures introduce certain assumptions that allow the use of a smaller model. In all cases there is a compromise between introducing too many assumptions, which limits accuracy, and too few, which increases model size. In the absence of *a priori* selection methods the following rules are useful.

The model should achieve good representation of inertia forces for the frequency range of interest. This is generally obtained by retaining a number of component *normal modes*. Boundary conditions used to compute these modes can be different than those of the full model (fixed [13], free [7], and loaded [15] interfaces have all been considered in different CMS methods). A usual alternative to retaining modes is the selection of internal DOFs in static and dynamic *condensation methods* [6], with more difficulties however for determining the bandwidth in which the model is valid [16].

The model should contain a good representation of the response to all loads of interests. In particular interface forces (for CMS methods), external loads, internal loads linked to model modifications [17] need to be considered. The loads of interest can be grouped into a general input shape matrix b_{Int} and one usually retains the *exact static responses* to these loads.

The static responses to the unit loads described by the input shape matrix b_{Int} are generally called *attachment modes* [1] and found by

$$[T_C] = [K]^{-1}[b_{Int}] \qquad (16)$$

In cases with rigid body modes, the stiffness matrix is singular and one uses attachment modes which correspond to the flexible response to the applied loads b_{Int}. These modes are computed by projecting the loads b_{Int} onto a subspace that is orthogonal to the inertia forces of rigid body modes ($M\phi_R$) and computing the associated response T_{CFlex} (which exists [3])

$$[K][T_{CFlex}] = \left([I] - [M\phi_R][\phi_R^T M\phi_R]^{-1}\right)[b_{Int}] \qquad (17)$$

The loads b_{Int} are generally applied on a subset q_{INT} of the DOFs and the static ($s = 0$) or dynamic ($s = j\omega$) responses to unit displacements q_{INT} (the static responses are called *constraint modes*) are used in place of attachment modes

$$T_{Cs} = \begin{bmatrix} I \\ -K(s)_{CC}^{-1} K(s)_{CInt} \end{bmatrix} \qquad (18)$$

Constraint modes correspond to a static (Guyan) condensation [5] on the interface DOFs and are used in many CMS methods (Craig-Bampton [13], branch mode analysis [15], etc.).

Inertia relief modes are defined by the static response to inertia forces of rigid body modes

$$T_R = \begin{bmatrix} 0 & 0 \\ 0 & K_{CC}^{-1} \end{bmatrix} [M\phi_R] \qquad (19)$$

It was shown [1] that the range of the constraint modes + inertia relief modes and the range of the attachment modes are identical. Attachment or constraint modes should thus be chosen based on computational consideration.

3.2. Generalized interface degrees of freedom.

To achieve a good representation of interface loads, CMS methods have generally assumed that independent loads or displacements could be applied to all boundary nodes. As a result, as many vectors as boundary DOFs are used. For plate or solid interfaces the number of modes considered can be quite large and this has been rapidly recognized as a being a significant limitation.

Constraint modes (18) can be generalized by defining a basis of interface deformations T_{Int} and computing

$$T_{CsG} = \begin{bmatrix} T_{Int} \\ -K_{CC}^{-1} K_{CInt} T_{Int} \end{bmatrix} = [T_{Cs}][T_{Int}] \qquad (20)$$

The generalized constraint modes are linear combinations of the standard ones, but do not imply the need to assemble the condensation matrix $K_{CC}^{-1} K_{CInt}$ which can be a significant computational advantage. Several methods can be considered (see examples in section 4.1) to select the basis T_{Int} but the idea is as always to find a basis that is representative of actual displacements. Attachment modes (16) can be similarly extended by considering a matrix b_{Int} with less columns than interface DOFs.

In a model reduction (15), these generalizations of the usual interface modes are associated to coordinates that can appropriately be named *generalized interface degrees of freedom*.

Generalized constraint modes have the significant advantage of being compatible for components with compatible meshes. (obviously if the same T_{Int} is used for both sides of an interface, it is possible to build global shape functions that verify the continuity constraints). Generalized attachment modes and approaches using different bases of interface deformations lead to incompatible models which are considered below.

3.3. Reduction and incompatible component models

The use of incompatible models is illustrated in figure 2. For the piece-wise linear and higher order polynomial shape functions shown, enforcement of continuity at all points leads to no displacement. Meaningful results can be achieved by enforcing continuity at a reduced set of intermediate points. The choice of intermediate points is an important issue. In the figure selecting the ● nodes rather than the ○ node limits the average gap. In the figure the ● nodes were obviously selected with care. For most arbitrary choices of two or more nodes, zero motion would be the only possibility. Such impossibility of motion, called locking, can result in poor predictions and must be taken into account.

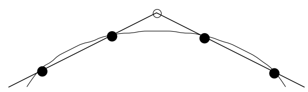

Fig. 2: Piece-wise linear and higher order polynomial shape functions with possible intermediate points ● or ○ for continuity enforcement.

Different factors can lead to incompatible models. Incompatibility of element shape functions is the most classical and the use of incompatible elements is widespread (see most textbooks on the finite element method). Incompatible meshes have been studied by a number of people (Refs. [18,2] for example) and the intermediate node approach shown in section 2.2 gives another possibility. In the present study, incompatibility of generalized constraint or attachment modes will be addressed. All types of incompatibility are clearly strongly related and methods can be extended to treat all cases. The problem is however important enough to justify the use of different point of views to achieve a better global understanding.

The point of view introduced in section 2.2 was to start with an obviously redundant set of boundary conditions on relative nodal displacements which for 2 substructures can be written in the form

$$\begin{bmatrix} c_{Int1} & -c_{Int2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0 \qquad (21)$$

For components reduced using bases T_1 and T_2 (of dimensions n_j by n_{Rj}), condition (21) becomes

$$\begin{bmatrix} c_{Int1} T_1 & -c_{Int2} T_2 \end{bmatrix} \begin{Bmatrix} q_{R1} \\ q_{R2} \end{Bmatrix} = 0 \qquad (22)$$

For compatible models, interface deformations of the two components are described by the same number of degrees of freedom (ranks n_{Int1} and n_{Int2} of $c_{Int1}T_1$ and $c_{Int2}T_2$) and this number is equal to the number n_{Int} of independent constraints (rank of $\begin{bmatrix} c_{Int1}T_1 & -c_{Int2}T_2 \end{bmatrix}$). When these conditions are not verified the model presents some level of locking.

Traditional CMS methods construct compatible models in many different ways. When using fixed interface normal modes, compatible models are easily found by using standard (18) or generalized (20) constraint modes. When using free or loaded interface normal modes, compatibility is achieved by using complete bases of constraint or attachment modes [7] or static extensions to other components [15]. Clearly, static extensions correspond to the use of generalized constraint modes linked to the considered interface deformations.

Generalized constraint modes provide an efficient approach to building compatible models but computational considerations may still be in favor of incompatible models. The difficulty is then to determine if locking is important and, if so, find an appropriate treatment (relaxation (just ignore) or simply penalization of certain constraints). By relaxing one looses the full verification of kinematic boundary conditions, so that the guarantee of monotonic convergence found for Ritz analyses is lost. In a number of cases however, this is the best approach. This topic is still very open and an example is analyzed in section 4.4.

4. IMPLEMENTATION ON A PRACTICAL EXAMPLE

4.1. Definition of the example

The methods introduced in previous sections will be analyzed for the example of a stiffened panel with a hole (see figure 3). The nominal configuration consists of a flat panel (0.60 x 0.30 x 0.007 m) with a hole of diameter 0.06m at its center and three stiffeners (height 0.03 m, thickness 0.007 m, the two edge stiffeners are place 0.12m away from the middle one).

The model is separated in three components a main piece (skin and middle stiffener) and two edge stiffeners. The model shown in figure 3 is composed of 356 QUAD4 elements, 466 nodes, 2796 DOFs of whom 252 duplicated interface DOFs (component models are initially appended as in (2) without taking the overlap of interface DOFs). A general form of loading is assumed so that the model does not take any symmetry into account. Computations are performed in the MATLAB environment [14].

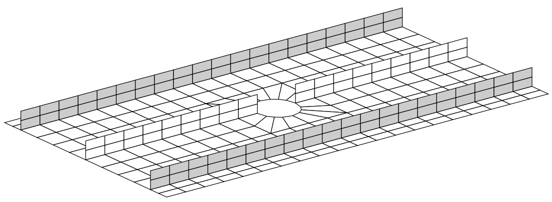

Fig. 3: Stiffened panel with a hole. The components are a main piece (skin and middle stiffener) and two edge stiffeners (shaded in gray).

One considers predictions of the structure with free boundary conditions. For this example, fixed or loaded boundary conditions would be more representative of industrial problems. This would however lead to ignore computational difficulties linked to the presence of rigid body modes, so that the free boundary conditions were considered here.

Comparisons will be done using three criteria (the relative frequency error, the Modal Assurance Criterion and the maximum singular value of the decomposition of the imbalance loads) defined below. The comparison is done on the first 10 flexible modes as in all considered cases the 6 rigid body modes are predicted exactly.

The first two criteria assume that the exact solution is known. The Modal Assurance Criterion (which measures the correlation between two modes)

$$MAC(\{\phi_1\},\{\phi_2\}) = \frac{\left(\{\phi_1\}^T\{\phi_2\}\right)^2}{\left(\{\phi_1\}^T\{\phi_1\}\right)\left(\{\phi_2\}^T\{\phi_2\}\right)} \tag{23}$$

is used to compare spatial properties of computed modes and to pair reduced model predictions and exact modes. The relative difference of frequencies for paired modes are then computed and used as a second criterion.

The third criterion s_{MAX} does not imply the knowledge of the exact solution and gives a single number a direct evaluation of errors in modeshape and frequencies. This criterion, introduced in Ref. [19], corresponds to the maximum singular value of a decomposition of the imbalance loads defined for a set of predicted modes (here the first 10 modes of the reduced models) by

$$b_j = \left[-\omega_j^2 M + K\right]\{\phi_j\} \tag{24}$$

s_{MAX} gives a measure of the maximum strain energy associated to the a unit combination of the imbalance loads. Since these loads are representative of the error done on the predicted normal modes, s_{MAX} gives a direct measure of model accuracy.

4.2. Reduction using generalized interface modes

The construction of generalized constraint or attachment modes is linked to the choice of a set of representative deformations of the interface or loads applied to the interface. The "representativity" is very much case dependent so that selection methods need to be introduced.

A first approach is to discretize the interface and use traditional finite element shape functions. For the example, interfaces are lines. Polynomial shape functions used for beams are thus appropriate. As usual there is a choice between *h* (geometrical division of the interface) or *p* (increase of the order of the considered polynomials) refinement. The *h* refinement is more flexible in terms of allowing complex geometry of the interface but links with convergence are not obvious.

For a given discretization, further reduction can be achieved by introducing arbitrary physical properties and computing global deformations. For example, two "modes" of an interface model corresponding to a beam discretization are shown in figure 4 (A-B).

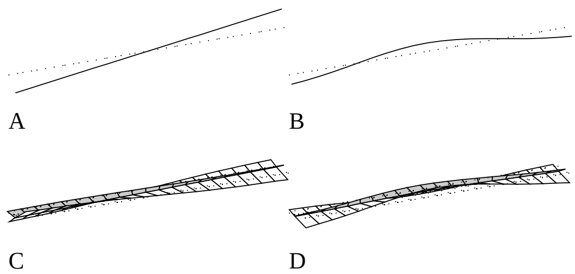

Fig. 4: A-B: interface deformations linked to the interpolation functions of standard beam elements. C-D: interface deformations linked to a local model.

Arbitrary interface models do not necessarily take physical properties of the systems into account. For example one can wonder whether it is more important to represent extension of the interface rather than flexion. Such decisions can be made by solving representative sub-problems.

The local model approach retains elements connected to the interface and computes modes of this model (low frequency modes correspond to deformations that have the lowest strain energy and are thus the most likely deformations of the interface [19]). For example, figure 4 C-D shows two modes of a local model of the interface.

The local model approach can obviously be extended by considering a large fraction of the system elements while keeping model size low by statically condensing interior DOFs not connected to the interface. In the limit one can consider a static condensation of the full model onto the interface DOFs and use generalized constraint modes that correspond to the low frequency modes of the associated reduced model. This approach was proposed by Craig and Chang [9] and one can show that it corresponds to an optimal selection of generalized constraint modes [19].

Shapes defined on interface DOFs can be seen as representative of interface deformations (leading to generalized constraint modes) or loads (leading to generalized attachment modes). As generalized attachment modes are generally not compatible, more vectors must be retained so that the gain linked to the use of generalized interface vectors becomes less obvious.

4.3. Predictions using generalized interface modes

To validate the interface mode construction methods introduced in section 4.2. Four models were considered. In all cases 20 flexible modes of the main piece with the interfaces with the edge stiffeners fixed are retained. The first frequency of the edge stiffeners with one side fixed is more than 1.5 times above the last frequency of interest so that no stiffener modes are retained. Different models are considered and qualities of the associated predictions are summarized in table 1.

Table 1: accuracy of normal mode predictions for different models. Mean relative error on frequency, minimum MAC (23) and maximum imbalance singular value s_{MAX}.

Model	size	$mean(\Delta\omega/\omega)$	$\min(MAC)$	s_{MAX} *10^{-3}
CB	272	0.44 %	99.02	1.30
CBP20	40	0.45 %	99.17	1.30
CBL	120	0.47 %	99.16	1.33
CBLP	40	0.87 %	66.41	1.52

For reference, the **CB** model defines one constraint mode for each of the 252 interface DOFs leading to a model of the main piece with 272 DOFs and two models of the stiffeners with 126 DOFs each. Interface continuity introduces 252 constraints which once eliminated lead to the classical Craig-Bampton [13] model with 272 DOFs.

The **CBP20** (Craig-Bampton with principal interface modes) retains the flexible modes of the main piece and the first 20 modes of the condensation of the model onto the interface DOFs. This model which has much fewer DOFs is almost as accurate. In the present case there are however significantly mode interface DOFs than interior DOFs. The computational costs of the CB and CBP models are thus very similar.

A reduction in the cost of the CBP20 model can be found through the use local models to construct a set of generalized constraint modes. Model **CBL** corresponds to the use of 50 generalized constraint modes with interface deformations given by modes of the local model shown in figure 4 C-D. The marginal deterioration of results (when compared with models CB and CBP) validates the approach. Model **CBLP** computes principal contributions of the basis retained in CBL. The deterioration of results is minor which again validates the approach (the low minimum MAC is linked to a recombination of closely spaced modes which does not imply large errors).

For the considered example, the use of interface shape functions based on a beam model poses a problem of compatibility with the elements. The relation between translations and rotations of rigid body modes are not the same for the standard beam model and the considered plate elements. Meaningful results are thus only obtained trough relatively complex extensions which fall beyond the purpose of this paper.

4.4. Continuity and accuracy : a trade-off for incompatible component models

The simple case of a model, where different numbers of principal constraint modes are retained for the main piece and the stiffeners, gives a good illustration of the trade-offs found for incompatible models. Measures of accuracy for the considered models are summarized in table 2.

A reference model **CBP15** is constructed by imposing all continuity constraints on an initial model with 20 fixed interface modes of the main piece and 15 principal constraint modes considered for all components. Since the same number of generalized constraint modes are retained for all components, the model is compatible. If other generalized constraint modes of the main piece are considered, the model becomes incompatible. An exact enforcement of interface continuity leads to eliminate these additional degrees of freedom (one keeps using model CBP15). The additional retained modes are fully locked and thus useless.

The simplest incompatible model approach is to leave those degrees of freedom completely free (no penalization). As shown in table 2, this can significantly improve results: model **CBP15+5** where five additional principal constraint modes of the main piece are retained is clearly much more accurate. (Once again close modal spacing explains the low minimum MAC while the more robust s_{MAX} indicator shows that the model is in fact quite good). Further analysis shows however that letting free just one of these generalized interface DOFs (the fourth by order of frequency) leads to a model (**CBP15+1**) which is almost as accurate as **CBP15+5**. Selection methods for the generalized interface DOFs to be retained would thus be useful.

When using unpenalized incompatible modes, monotonic convergence properties linked to the use of a Ritz type approach are lost. Here, the use of additional DOFs linked to relatively high frequency principal constraint modes does not permit low energy deformations that would significantly decrease predicted modal frequencies but there is no general guarantee.

For compatible meshes (and, with extensions, for incompatible meshes) a simple penalization can be found by taking into account the overlap of interface nodes (considering that the interface deforms but assuming zero motion of the interior DOFs of other components). This approach guarantees monotonic convergence, but model **CBP15+5P** which uses this penalization shows minor improvements over model CBP15. Here deforming the interface with no interior deformation is associated to too much strain energy to allow any motion. This shows that locking and over-stiffening are strongly related.

Table 2: accuracy of normal mode predictions for different models. Mean relative error on frequency, minimum MAC (23) and maximum imbalance singular value s_{MAX}.

Model	size	$mean(\Delta\omega/\omega)$	$\min(MAC)$	s_{MAX} *10^{-3}
CBP15	35	43.36 %	43.44	10.92
CBP15+5	40	1.17 %	69.22	1.30
CBP15+1	36	2.10 %	69.49	1.30
CBP15+5P	40	36.13 %	55.74	9.46
CBP20	40	0.45 %	99.17	1.30

Another penalization would be to consider static extensions of the principal constraint mode which comes back to the use of global principal constraint modes (the very accurate model **CBP20**). This highlights the fact that penalization can be seen as the extension of incompatible modes to all or part of neighboring components.

5. Conclusion

A general framework for the treatment CMS problems with arbitrary interface deformations was introduced. Interface displacement continuity conditions appeared as generalized kinematic boundary conditions and interface force equilibrium conditions lead to model reduction through static condensation. For the prediction of the coupled response, which corresponds to the resolution of a constrained problem, numerical issues linked to the direct method (elimination of the constraints) were addressed.

As the use of arbitrary interface deformations can be considered, extensions to the traditional attachment or constraint modes clearly become interesting. Through a fairly complex example, practicality of the direct constraint elimination approach and of the proposed generalized constraint modes was demonstrated. For this and other examples, the proposed methods were found to be very robust.

The use of incompatible models introduces locking phenomena which must be properly treated. The consideration of locking for incompatible component models gives a significantly different perspective on locking, a phenomenon previously studied for incompatible elements and meshes. Finally, the construction constraint penalization through local or global extensions shows a great potential.

6. References

[1] Craig, R. R. Jr., “A Review of Time-Domain and Frequency Domain Component Mode Synthesis Methods,” *Modal Analysis*, 1987, **2**-2, pp. 59-72

[2] Farhat, C., Géradin, M., “A Hybrid Formulation of a Component Mode Synthesis Method,” *33rd SDM Conference, AIAA Paper 92-2383-CP*, 1992, pp. 1783-1796

[3] Géradin, M., Rixen, D., *Mechanical Vibrations. Theory and Application to Structural Dynamics.* John Wiley & Wiley and Sons, 1994, Masson, Paris, 1993

[4] Jen, C.W., Johnson, D.A., Dubois, F., “Numerical modal analysis of structures based on a revised substructure synthesis approach,” *Journal of Sound and Vibration*, **180**-3, 1995, pp. 185-203

[5] Guyan, R.J., “Reduction of Mass and Stiffness Matrices,” *AIAA Journal*, 1965, **3**

[6] O'Callahan, J.C., “Comparison of Reduced Model concepts,” *IMAC*, 1990

[7] Rubin, S., “Improved Component-Mode Representation for Structural Dynamic Analysis,” *AIAA Journal*, 1975, **13**-8

[8] Craig, R.R., Chang, C.J., “On the use of attachment modes in substructure coupling for dynamic analysis,” *AIAA Paper 77-405-CP*, 1977, pp. 89-99

[9] Craig, R.R., Chang, C.J., “Substructure Coupling for Dynamic Analysis and Testing,” NASA CR-2781, 1977

[10] Farhat, C., Roux, F.-X., “Implicit parallel processing in structural mechanics,” *Computational Mechanics Advances*, Elsevier Science B.V., Amsterdam, 1994

[11] Zienkiewicz, O.C., Taylor, R.L., *The Finite Element Method*, MacGraw-Hill, 1989

[12] Ohayon, R., Sampaio, R., Soize, C., “Dynamic Substructuring of Damped Structures Using Singular Value Decomposition,” Submitted to *Journal of Applied Mechanics*, 1995

[13] Craig, R.R. Jr., Bampton, M.C., “Coupling of Substructures for Dynamic Analyses,” *AIAA Journal*, 1968, **6**-7

[14] Balmès, E., *Structural Dynamics Toolbox 2.0*, 1995, (A toolbox for MATLAB™, Scientific Software Group (France), e-mail: info@ssg.fr)

[15] Gladwell, G.M.L., “Branch mode analysis of vibrating systems,” *J. Sound Vib.*, 1964, **1**, pp. 41-59

[16] Bouhaddi, N., Cogan, S., Fillod, R., “Dynamic Substructuring by Guyan Condensation, Selection of the Master DOF,” *IMAC*, 1992

[17] Balmès, E., “Parametric families of reduced finite element models. Theory and applications.,” *IMAC*, 1995

[18] Bernadi, C., Maday, Y., Patera, A.T., “Domain decomposition by the mortar element method,” Laboratoire d'analyse numérique, Paris VI Univ., Raport 92013

[19] Balmès, E., “Optimal Ritz vectors for component mode synthesis using the singular value decomposition.” Submitted to the *AIAA Journal*, August 1995